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Weyl's type estimates on the eigenvalues of critical Schrödinger operators using improved Hardy–Sobolev inequalities

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Abstract
Motivated by the work (Karachalios N I 2008 *Lett. Math. Phys.* **83** 189–99), we present explicit asymptotic estimates on the eigenvalues of the critical Schrödinger operator, involving inverse-square potential, based on improved Hardy–Sobolev-type inequalities.

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1. Introduction

In this paper, we present explicit asymptotic estimates of the critical Schrödinger operator, based on recent results on improved Hardy–Sobolev-type inequalities, motivated by the work [5] where Weyl' s type estimates where obtained by using improved Hardy inequalities.

More precisely, in [5], the author considered the eigenvalue problem for the critical Schrödinger operator $-\Delta - V$,

$$\begin{aligned} &-\Delta u - V(x)u = \lambda u, \qquad \text{in} \quad \Omega, \\ &u = 0, \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, containing the origin and

$$V(x) = \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}.$$

The term critical is due to the fact that the constant $C^* = (N-2)^2/4$ is the optimal constant of the Hardy inequality

$$C \int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2\,\mathrm{d}x, \qquad \text{for all} \quad u \in C_0^\infty(\Omega), \Omega \subseteq \mathbb{R}^N, \ N \geqslant 3, \tag{1.2}$$

which is not attained in $H_0^1(\Omega)$. Based on the following improved Hardy inequalities (see [3, 10]),

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x \geqslant c(q,\Omega)\left(\int_\Omega u^q\,\mathrm{d}x\right)^{1/q}, 1 \leqslant q < \frac{2N}{N-2}, \tag{1.3}$$

which holds for every $u \in C_0^\infty(\Omega)$, the author in [5] proved the following Weyl's type estimates for the eigenvalues of (1.1)

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded domain containing the origin. Assume further that*

$$\frac{2N}{N+2} < q < 2. \tag{1.4}$$

The eigenvalues of the problem (1.1) with potential $V(x) = (N-2)^2/4|x|^2$ satisfy Weyl's estimate

$$\lambda_j \geqslant C(q,\Omega)\mathrm{e}^{-1}\mu_N(\Omega)^{-\frac{Nq-2N+2q}{Nq}} j^{\frac{Nq-2N+2q}{Nq}}, \qquad j \to \infty, \tag{1.5}$$

where $\mu_N(\Omega)$ denotes the Lebesgue measure of Ω in $\mathbb{R}^N$.

In this work we improve these Weyl's type estimates by using improved Hardy–Sobolev inequalities; theorem A in [4] states that

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, containing the origin and $D > D_0 := \sup_{x\in\Omega}|x|$. Then, there exists a positive constant C_{HS}, such that*

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x \geqslant \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x + C_{HS}\left(\int_\Omega |u|^{\frac{2N}{N-2}}\left(-\log\left(\frac{|x|}{D}\right)\right)^{-\frac{2(N-1)}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{N}} \tag{1.6}$$

for all $u \in C_0^\infty(\Omega)$.

This estimate is sharp in the sense that the power $-\frac{2(N-1)}{N-2}$ in the logarithm cannot be replaced by a bigger power. We note that in the radial case, i.e. where $\Omega = B_R$ is the open ball in $\mathbb{R}^N$, $N \geqslant 3$, of radius R centered at the origin and $u \in C_0^\infty(B_R\backslash\{0\})$ is a radially symmetric function, the same result was proved in [11], with the use of a Caffarelli–Kohn–Nirenberg inequality and in [15] with the use of Bliss' inequality. From the discussion in [4, 11], it is clear that the nature of (1.6) depends on the distance of D from D_0; for instance in the case where $D = D_0$, the author in [11] proved that the inequality cannot hold in the nonradial case.

The best constant C_{HS} in (1.6), as it was obtained in [1], is as follows.

Lemma 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, containing the origin and $D_0 = \sup_{x\in\Omega}|x|$. If*

$$D \geqslant D_0\,\mathrm{e}^{\frac{1}{N-2}}, \tag{1.7}$$

then the best constant in (1.6) is given by

$$C_{HS} := S(N)\,(N-2)^{-\frac{2(N-1)}{N}}, \tag{1.8}$$

where $S(N)$ is the best constant in the Sobolev inequality in $\mathbb{R}^N$ and there exists no minimizer in $H_0^1(\Omega)$.

As is well known, see [2, 9, 12], the best constant in the Sobolev inequality in $\mathbb{R}^N$,

$$\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x \geqslant S\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,\mathrm{d}x\right)^{\frac{N-2}{N}}, \tag{1.9}$$

is

$$S(N) = \frac{N(N-2)}{4}|\mathbb{S}_N|^{2/N} = 2^{2/N}\pi^{1+1/N}\Gamma\left(\frac{N+1}{2}\right)^{-2/N}, \tag{1.10}$$

where $\mathbb{S}_N$ is the area of the N-dimensional unit sphere and the extremal functions are

$$\psi_{\mu,\nu}(|x|) = (\mu^2 + \nu^2|x|)^2)^{-(N-2)/2}, \qquad \mu \neq 0, \quad \nu \neq 0. \tag{1.11}$$

The same best constant for (1.6) was obtained in [15] for the radial case of (1.6) i.e. assuming only radial functions defined on $\Omega = B_R$ and $D = R$.
In this case the minimizers are given by

$$\phi_{m,n}(|x|) = |x|^{-\frac{N-2}{2}}\psi_{m,n}\left(\left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{1}{N-2}}\right), \qquad x \in B_R\backslash\{0\}, \quad \phi_{m,n}|_{\partial B_R} = 0. \tag{1.12}$$

where $\psi_{m,n}$ are the extremals of the Sobolev inequality.

2. Explicit Weyl's type estimates

Using (1.6), lemma 1.1 and the arguments of [8] we may prove that

Theorem 2.1. *Assume that $D \geqslant D_0\,\mathrm{e}^{\frac{1}{N-2}}$. Then, the eigenvalues of the problem (1.1) with potential $V(x) = (N-2)^2/4|x|^2$ satisfy Weyl's estimate*

$$\begin{aligned}\lambda_j &\geqslant C_{HS}\,\mathrm{e}^{-1}\,||X^{-\frac{N-1}{2}}||_{L^2(\Omega)}^{-\frac{4}{N}}\,j^{\frac{2}{N}} \\ &\geqslant S(N)\,(N-2)^{-\frac{2(N-1)}{N}}\,\mathrm{e}^{-1}\,||X^{-\frac{N-1}{2}}||_{L^2(\Omega)}^{-\frac{4}{N}}\,j^{\frac{2}{N}},\end{aligned} \tag{2.1}$$

where $X := \left(-\log\left(\frac{|x|}{D}\right)\right)^{-1}$.

Proof. We adapt the arguments of [5] in our case. Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded domain containing the origin. We define the operator

$$\mathcal{K}_0 = -\Delta - \frac{(N-2)^2}{4|x|^2}$$

with the domain of definition $D(\mathcal{K}_0) = C_0^\infty(\Omega)$ and its Friedrich's extension $\mathcal{K} : D(\mathcal{K}) \to L^2(\Omega)$, with its domain defined as

$$D(\mathcal{K}) := \left\{u \in H(\Omega) \;:\; -\Delta u - \frac{(N-2)^2}{4|x|^2}u \in L^2(\Omega)\right\},$$

which is a nonnegative self-adjoint operator on $L^2(\Omega)$. Then as it was proved in [14]: there exists a complete orthonormal basis $\{\phi_j\}_{j\geqslant 1}$ of $L^2(\Omega)$ consisting of eigenfunctions of $\mathcal{K}$ with the eigenvalue sequence

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to \infty, \qquad \text{as} \quad j \to \infty. \tag{2.2}$$

The operator $\mathcal{K}$ being non-negative and self-adjoint in $L^2(\Omega)$ gives rise to the semigroup of operators $\mathrm{e}^{-\mathcal{K}t}$ for every $t > 0$, possessing an integral kernel $k(x, y, t) > 0$ for all $(x, y, t) \in \Omega \times \Omega \times (0, \infty)$. The operator $\mathcal{K}$ has compact resolvent; thus, $k(x, y, t)$ can also be represented as

$$k(x, y, t) = \sum_{n=1}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y). \tag{2.3}$$

We quote that $k(x, y, t)$ solves the problem

$$\partial_t k(x,y,t) - \Delta_y k(x,y,t) - \frac{(N-2)^2}{4|y|^2} k(x,y,t) = 0, \quad \text{in} \quad \Omega \times \Omega \times (0,\infty), \quad t > 0,$$
$$k(x,y,t) > 0, \quad \text{in} \quad \Omega \times \Omega \times (0,\infty),$$
$$k(x,y,t) = 0, \quad \text{in} \quad \partial\Omega \times \partial\Omega \times (0,\infty). \tag{2.4}$$

Since $\{\phi_j\}_{j\geqslant 1}$ is an orthonormal basis of $L^2(\Omega)$, it follows that

$$f(t) := \sum_{n=1}^{\infty} \exp(-2\lambda_n t) = \int_\Omega \int_\Omega k^2(x,y,t)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.5}$$

Applying Hölder's inequality and the usual 1-trick, we get that

$$f(t) = \int_\Omega \int_\Omega \left[X^{\frac{2(N-1)}{N-2}} \left(\frac{|y|}{D}\right) k^{p^*}(x,y,t) \right]^{\frac{1}{p^*-1}} \cdot \left[X^{-\frac{2(N-1)}{(N-2)(p^*-2)}} \left(\frac{|y|}{D}\right) k(x,y,t) \right]^{\frac{p^*-2}{p^*-1}} \mathrm{d}x\,\mathrm{d}y \tag{2.6}$$

$$\leqslant \int_\Omega \left[\int_\Omega X^{\frac{2(N-1)}{N-2}} \left(\frac{|y|}{D}\right) k^{p^*}(x,y,t)\,\mathrm{d}y \right]^{\frac{1}{p^*-1}} \cdot \left[\int_\Omega X^{-\frac{2(N-1)}{(N-2)(p^*-2)}} \left(\frac{|y|}{D}\right) k(x,y,t)\,\mathrm{d}y \right]^{\frac{p^*-2}{p^*-1}} \mathrm{d}x \tag{2.7}$$

$$\leqslant \left[\int_\Omega \left(\int_\Omega X^{\frac{2(N-1)}{N-2}} \left(\frac{|y|}{D}\right) k^{p^*}(x,y,t)\,\mathrm{d}y \right)^{\frac{2}{p^*}} \mathrm{d}x \right]^{\frac{p^*}{2(p^*-1)}} \tag{2.8}$$

$$\cdot \left(\int_\Omega \mathcal{Q}^2(x,t)\,\mathrm{d}x \right)^{\frac{p^*-2}{2(p^*-1)}}, \tag{2.9}$$

where $\mathcal{Q}$ denotes the function

$$\mathcal{Q}(x,t) = \int_\Omega X^{-\frac{2(N-1)}{(N-2)(p^*-2)}} \left(\frac{|y|}{D}\right) k(x,y,t)\,\mathrm{d}y = \int_\Omega X^{-\frac{N-1}{2}} \left(\frac{|y|}{D}\right) k(x,y,t)\,\mathrm{d}y. \tag{2.10}$$

Observe that the function $X^{-\frac{N-1}{2}}$ belongs to $L^2(\Omega)$; more precisely,

$$||X^{-\frac{N-1}{2}}||^2_{L^2(\Omega)} = \int_\Omega X^{-(N-1)} \left(\frac{|y|}{D}\right) \mathrm{d}y \leqslant \int_{B_D} X^{-(N-1)} \left(\frac{|y|}{D}\right) \mathrm{d}y$$
$$\leqslant N\omega_N \int_0^D r^{N-1} \left(-\log\frac{r}{D}\right)^{N-1} \mathrm{d}r$$

and after some proper change of variables, we have that

$$||X^{-\frac{N-1}{2}}||^2_{L^2(\Omega)} \leqslant N\,\omega_N\,D^N \int_0^1 t^{N-1}\,(-\log t)^{N-1}\,\mathrm{d}t$$
$$= N\,\omega_N\,D^N \int_0^\infty s^{N-1}\,\mathrm{e}^{-Ns}\,\mathrm{d}s = \omega_N\,D^N\,\frac{(N-1)!}{N^{N-1}}. \tag{2.11}$$

Note that the equality holds in the case where Ω is a sphere with center the origin and radius D. Moreover, (2.3) and the fact that $\{\phi_j\}_{j\geqslant 1}$ are orthonormal in $L^2(\Omega)$ imply that

$$\mathcal{Q}(x,0) = X^{-\frac{N-1}{2}} \left(\frac{|x|}{D}\right).$$

Thus, $\mathcal{Q}(x,t)$ is the solution of the Cauchy–Dirichlet problem:

$$\partial_t \mathcal{Q}(x,t) - \Delta_x \mathcal{Q}(x,t) - \frac{(N-2)^2}{4|x|^2}\mathcal{Q}(x,t) = 0, \qquad \text{in} \quad \Omega \times (0,\infty), \tag{2.12}$$

$$\begin{aligned} &\mathcal{Q}(x,0) = X^{-\frac{N-1}{2}}\left(\frac{|x|}{D}\right), && \text{for} \quad x \in \Omega, \\ &\mathcal{Q}(x,t) = 0, && \text{in} \quad \partial\Omega \times (0,\infty). \end{aligned} \tag{2.13}$$

Multiplying the heat equation (2.12) by $\mathcal{Q}$ in the $L^2(\Omega)$-inner product, we get the energy equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\mathcal{Q}(t)||^2_{L^2(\Omega)} + ||\mathcal{Q}(t)||^2_{H_\mu(\Omega)} = 0.$$

The energy equation above, combined with (2.13), implies that

$$||\mathcal{Q}(t)||^2_{L^2(\Omega)} \leqslant ||\mathcal{Q}(0)||^2_{L^2(\Omega)} = \left|\left|X^{-\frac{N-1}{2}}\right|\right|^2_{L^2(\Omega)}. \tag{2.14}$$

Now, by inserting (2.14) into (2.6), the inequality

$$f^{\frac{2(p^*-1)}{p^*}}(t) \leqslant ||X^{-\frac{N-1}{2}}||^{\frac{2(p^*-2)}{p^*}}_{L^2(\Omega)} \int_\Omega \left(\int_\Omega X^{\frac{2(N-1)}{N-2}}\left(\frac{|y|}{D}\right) k^{p^*}(x,y,t)\,\mathrm{d}y\right)^{\frac{2}{p^*}} \mathrm{d}x \tag{2.15}$$

follows. The right-hand side of (2.15) can be estimated further, by applying the improved Hardy–Sobolev inequality (1.6). Thus,

$$f^{\frac{2(p^*-1)}{p^*}}(t) \leqslant \frac{||X^{-\frac{N-1}{2}}||^{\frac{2(p^*-2)}{p^*}}_{L^2(\Omega)}}{C_{HS}} \int_\Omega\int_\Omega \left(\left|\nabla_y k(x,y,t)\right|^2 - \frac{(N-2)^2}{4|y|^2}k^2(x,y,t)\right)\mathrm{d}y\,\mathrm{d}x. \tag{2.16}$$

On the other hand, by (2.3) and equation (2.4), we deduce that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}f(t) &= \int_\Omega\int_\Omega k(x,y,t)\,\partial_t k(x,y,t)\,\mathrm{d}x\,\mathrm{d}y \\ &= -2\int_\Omega\int_\Omega \left(\left|\nabla_y k(x,y,t)\right|^2 - \frac{(N-2)^2}{4|y|^2}k^2(x,y,t)\right)\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17), we get the differential inequality

$$\frac{\mathrm{d}f(t)}{f^{\frac{2(p^*-1)}{p^*}}(t)} \leqslant \frac{-2C_{HS}}{||X^{-\frac{N-1}{2}}||^{\frac{2(p^*-2)}{p^*}}_{L^2(\Omega)}}\mathrm{d}t,$$

which if integrated with respect to time yields

$$f(t) = \sum_{n=1}^{\infty}\exp(-2\lambda_n t) \leqslant \left[\frac{p^*}{2C_{HS}(p^*-2)}\right]^{\frac{p^*}{p^*-2}} ||X^{-\frac{N-1}{2}}||^2_{L^2(\Omega)} t^{-\frac{p^*}{p^*-2}}. \tag{2.18}$$

Setting

$$t = \frac{p^*}{2(p^*-2)}\frac{1}{\lambda_j},$$

in (2.18), we conclude with the estimate

$$j\exp\left(-\frac{p^*}{p^*-2}\right) \leqslant \sum_{n=1}^{\infty}\exp\left[\frac{-\lambda_n p^*}{\lambda_j(p^*-2)}\right] \leqslant C_{HS}^{-\frac{p^*}{p^*-2}}\lambda_j^{\frac{p^*}{p^*-2}}||X^{-\frac{N-1}{2}}||^2_{L^2(\Omega)}. \tag{2.19}$$

From this inequality and the fact that $(p^* - 2)/p^* = 2/N$, we conclude that

$$\lambda_j \geqslant C_{HS}\, \mathrm{e}^{-1} \|X^{-\frac{N-1}{2}}\|_{L^2(\Omega)}^{-\frac{4}{N}} j^{\frac{2}{N}}.$$

Finally, (1.8) implies the estimate (2.1). □

Remark 2.1. Weyl's estimate for the eigenvalue problem (see [8, theorem 1.2])

$$\begin{aligned} -\Delta u &= \mu u, && \text{in} \quad \Omega, \\ u &= 0, && \text{in} \quad \partial\Omega, \end{aligned} \tag{2.20}$$

is the following:

$$\mu_j \geqslant S(N)\, \mathrm{e}^{-1}\, \mu_N(\Omega)^{-\frac{2}{N}}\, j^{\frac{2}{N}}, \tag{2.21}$$

where $\mu_N(\Omega)$ denotes the Lebesgue measure of Ω in $\mathbb{R}^N$ and $S(N)$, given by (1.10), is the best constant in the Sobolev's inequality (see 1.9). We want to compare the estimates (2.21) and (2.1). We claim that

$$(N-2)^{-\frac{2(N-1)}{N}} \|X^{-\frac{N-1}{2}}\|_{L^2(\Omega)}^{-\frac{4}{N}} < \mu_N(\Omega)^{-\frac{2}{N}},$$

or

$$(N-2)^{(N-1)} \|X^{-\frac{N-1}{2}}\|_{L^2(\Omega)}^2 > \mu_N(\Omega). \tag{2.22}$$

For the proof of (2.22), we observe that

$$\begin{aligned} \|X^{-\frac{N-1}{2}}\|_{L^2(\Omega)}^2 &= \int_\Omega \left(-\log\left(\frac{|x|}{D}\right)\right)^{N-1} \mathrm{d}x > \left(-\log\left(\frac{D_0}{D}\right)\right)^{N-1} \int_\Omega \mathrm{d}x \\ &> \left(\log\left(\frac{D}{D_0}\right)\right)^{N-1} \mu_N(\Omega), \end{aligned}$$

and from (1.7) we have that

$$\|X^{-\frac{N-1}{2}}\|_{L^2(\Omega)}^2 > (N-2)^{-(N-1)} \mu_N(\Omega).$$

Thus, (2.22) is proved. As it was expected the eigenvalues of (1.1) are generally smaller than those of (2.20). We note also the results of [7] concerning the principal eigenvalue.

Using theorem 2.1, we may improve the results of [6] concerning the asymptotic behavior of the corresponding to (1.1) parabolic equation:

$$\phi_t - \nu\Delta\phi - \nu V(x)\phi + f(\phi) = 0, \qquad \nu > 0, \qquad \text{in} \quad \Omega,\ t > 0, \tag{2.23}$$

$$\phi(x,0) = \phi_0(x), \qquad \text{for} \quad x \in \Omega, \tag{2.24}$$

$$\phi(x,t) = 0 \qquad \text{in} \quad \partial\Omega,\ t > 0, \tag{2.25}$$

where the reaction term $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of odd degree with a positive leading coefficient

$$f(s) = \sum_{i=0}^{2\gamma-1} b_i s^i, \qquad b_{2\gamma-1} > 0. \tag{2.26}$$

We denote by k the positive number for which

$$f'(s) \geqslant -k, \qquad \text{for any} \quad s \in \mathbb{R}.$$

More precisely, in [6] using theorem 1.1, it was proved that

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded domain containing the origin. The initial-boundary value problem (2.23)–(2.24)–(2.25) with nonlinearity (2.26) defines a semiflow $\tilde{\mathcal{S}}(t) : L^2(\Omega) \to L^2(\Omega)$ possessing a global attractor $\tilde{\mathcal{A}}$. There exists a constant $C_1(q, \Omega, N) > 0$ such that $\dim_H \tilde{\mathcal{A}} \leqslant \tilde{d}_0$ with*

$$\tilde{d}_0 = C_1(q, \Omega, N)^{-\frac{Nq}{Nq-2N+2q}} \left(\frac{\kappa}{\nu}\right)^{\frac{Nq}{Nq-2N+2q}} \mu_N(\Omega). \tag{2.27}$$

However, the dimension of the attractor, using theorem 2.1 and the arguments of [6], may be estimated as

Lemma 2.1. *The dimension of the global attractor $\tilde{\mathcal{A}}$, obtained in theorem 2.2, satisfies $\dim_H \tilde{\mathcal{A}} \leqslant \tilde{d}$, with*

$$\tilde{d} = \left(\frac{\kappa}{\nu}\right)^{\frac{N}{2}} \mathrm{e}^{N/2} \left(\frac{N+2}{N}\right)^{N/2} C_{HS}^{-N/2} ||X^{-\frac{N-1}{2}}||^2_{L^2(\Omega)},$$

where C_{HS} is given in (1.8) and an estimate of $||X^{-\frac{N-1}{2}}||_{L^2(\Omega)}$ may be found in (2.11) and in (2.22).

Finally, we treat with the critical eigenvalue problem

$$\begin{aligned} &-\nabla\cdot(|x|^{-2m}\nabla u) - V(x)u = \lambda u, && \text{in} \quad \Omega, \\ &u = 0, && \text{in} \quad \partial\Omega, \end{aligned} \tag{2.28}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, containing the origin and

$$V(x) = \left(\frac{N-2m-2}{2}\right)^2 \frac{1}{|x|^{2m-2}}, \qquad 2 \leqslant m+2 \leqslant N.$$

The term critical is due to the fact that the constant $C^* = (N-2m-2)^2/4$ is the optimal constant of the Hardy-type inequality (e.g. see [13]):

$$\int_\Omega \frac{|\nabla u|^2}{|x|^{2m}}\,\mathrm{d}x \geqslant \left(\frac{N-2m-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+2}}\,\mathrm{d}x. \tag{2.29}$$

It is also clear that the improved Hardy–Sobolev inequality for this case is

$$\begin{aligned} &\int_\Omega \frac{|\nabla u|^2}{|x|^{2m}}\,\mathrm{d}x - \left(\frac{N-2m-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+2}}\,\mathrm{d}x \\ &\quad \geqslant C_{HS}\left(\int_\Omega |x|^{-\frac{2N}{N-2}m} |u|^{\frac{2N}{N-2}} \left(-\log\left(\frac{|x|}{D}\right)\right)^{-\frac{2(N-1)}{N-2}} \mathrm{d}x\right)^{\frac{N-2}{N}}, \end{aligned} \tag{2.30}$$

where $D \geqslant D_0\,\mathrm{e}^{\frac{1}{N-2}}$. The best constant C_{HS} is given by (1.8). Then, by standard arguments, we may prove that there exists a complete orthonormal basis $\{\phi_j\}_{j\geqslant 1}$ of $L^2(\Omega)$ consisting of eigenfunctions of (2.28) with the eigenvalue sequence

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to \infty, \quad \text{as} \quad j \to \infty. \tag{2.31}$$

Following the same arguments of theorem 2.1, we have that

Theorem 2.3. *Assume that $D \geqslant D_0\,\mathrm{e}^{\frac{1}{N-2}}$. Then, the eigenvalues of the problem (2.28) satisfy Weyl's estimate*

$$\begin{aligned} \lambda_j &\geqslant C_{HS}\,\mathrm{e}^{-1}\,|||x|^{\frac{Nm}{2}} X^{-\frac{N-1}{2}}||^{-\frac{4}{N}}_{L^2(\Omega)}\, j^{\frac{2}{N}} \\ &\geqslant S(N)\,(N-2)^{-\frac{2(N-1)}{N}}\,\mathrm{e}^{-1}\,|||x|^{\frac{Nm}{2}} X^{-\frac{N-1}{2}}||^{-\frac{4}{N}}_{L^2(\Omega)}\, j^{\frac{2}{N}}, \end{aligned} \tag{2.32}$$

where $X := \left(-\log\left(\frac{|x|}{D}\right)\right)^{-1}$.

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